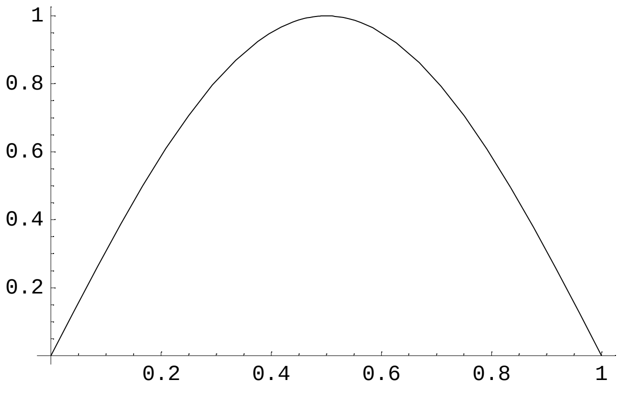
Rayleigh, Ritz-Galerkin

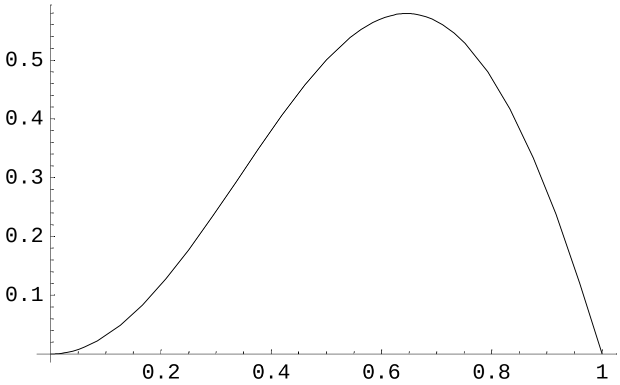
1. Rayleigh - Quotiont berechnen mit Versuchsfunktionen

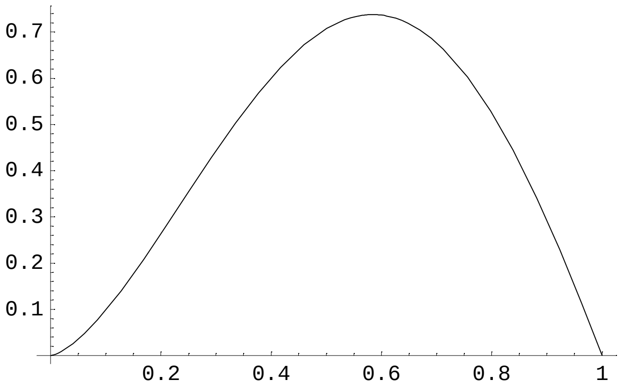
```
xL = 1;
R[y_, x1_] := -NIntegrate[y''[x]  y[x], {x, 0, x1}]/
                  NIntegrate[ x y[x]^2, {x, 0, x1}];
y[x_, 1 ] := Sin[(Pi)/xL x];
y[x_, 2 ] := x Sin[(Pi)/xL x];
y[x_, 3 ] := Sqrt[x] Sin[(Pi)/xL x];
y[x_, 4 ] := Sqrt[xL - x] Sin[(Pi)/xL x];
y[x_, 5 ] := x(x - xL);
y[x_, 6 ] := x (x - xL)^2;
y[x_, 7 ] := x^2 (x - xL);
y[x_, 8 ] := x^2 (x - xL)^2;
y[x_, 9 ] := x^3 (x - xL)^3;
y[x_, 10] := x^4 (x - xL)^4;
Table[y[x_] := y[x, k]; {k, y[x], R[y, xL]}, {k, 1, 10}] // MatrixForm
```

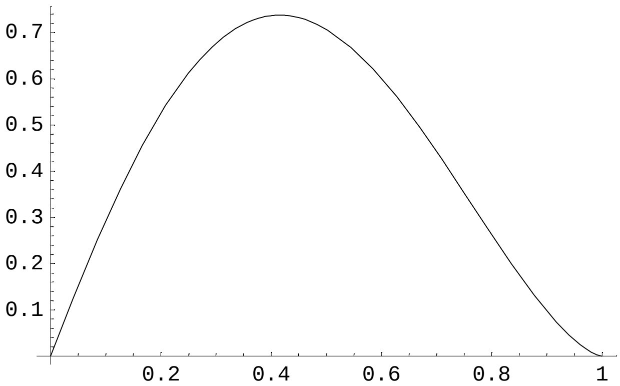
1	$\mathrm{Sin}[\pi x]$	19.7392
2	$x\,\mathrm{Sin}[\pi x]$	21.7797
3	$\sqrt{x}\,\mathrm{Sin}[\pi x]$	19.6135
4	$\sqrt{1-x}\,\mathrm{Sin}[\pi x]$	25.5109
5	$(-1+x)\,x$	20.
6	$(-1+x)^2\,x$	37.3333
7	$(-1+x)\,x^2$	22.4
8	$(-1+x)^2\,x^2$	24.
9	$(-1+x)^3\,x^3$	31.2
10	$(-1+x)^4\,x^4$	38.8571

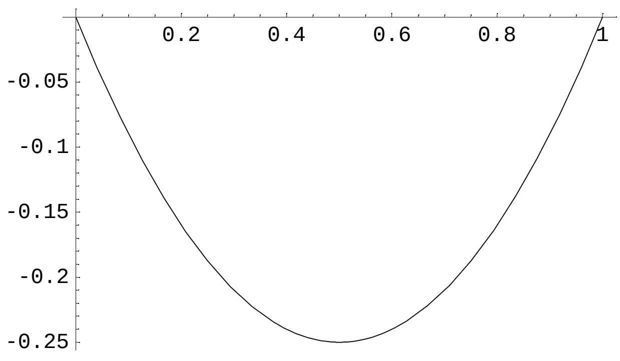
```
Table[Plot[y[x,s],{x,0,xL}],{s,1,10}];
```

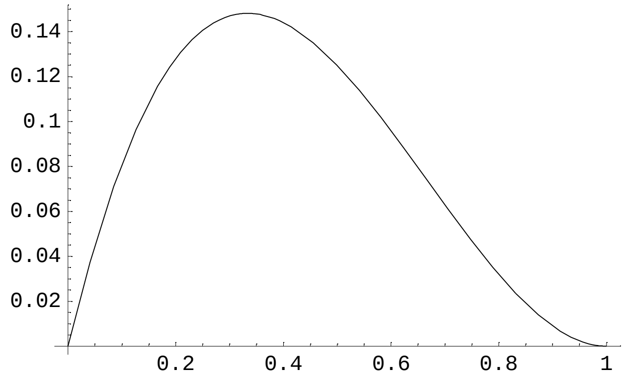


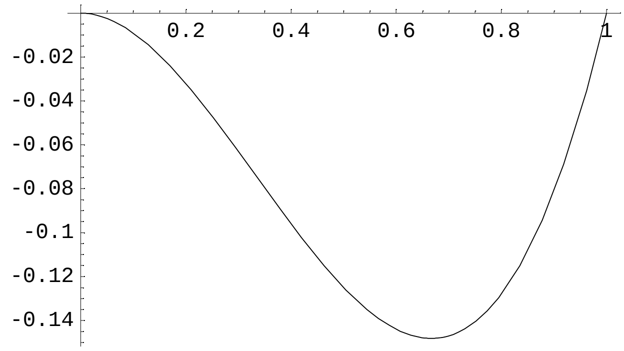

0.5
0.4
0.3
0.2
0.1
0.2 0.4 0.6 0.8 1

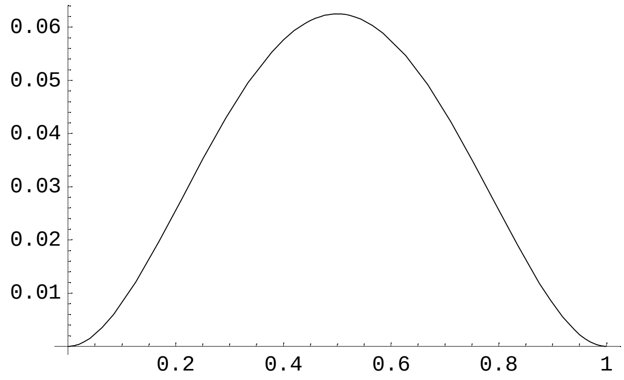

0.7
0.6
0.5
0.4
0.3
0.2
0.1
0.2 0.4 0.6 0.8 1

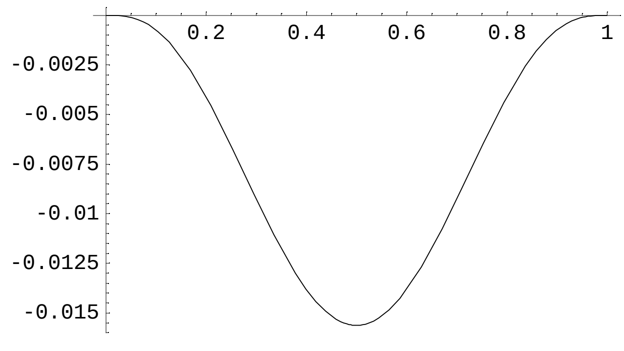

0.7
0.6
0.5
0.4
0.3
0.2
0.1
0.2 0.4 0.6 0.8 1

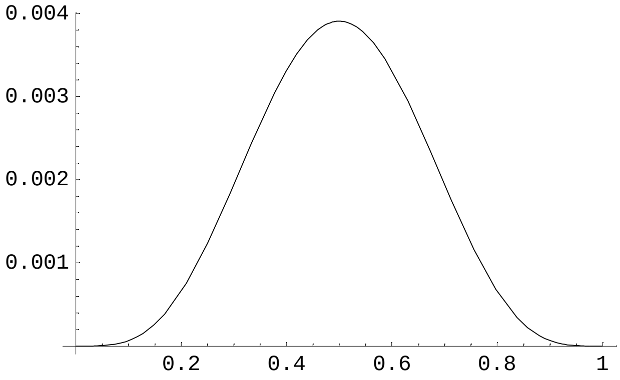

0.2 0.4 0.6 0.8 1
-0.05
-0.1
-0.15
-0.2
-0.25


0.14
0.12
0.1
0.08
0.06
0.04
0.02
0.2
0.4
0.6
0.8
1


0.2
0.4
0.6
0.8
1
-0.02
-0.04
-0.06
-0.08
-0.1
-0.12
-0.14


0.06
0.05
0.04
0.03
0.02
0.01
0.2
0.4
0.6
0.8
1


0.2
0.4
0.6
0.8
1
-0.0025
-0.005
-0.0075
-0.01
-0.0125
-0.015



2. Eigenwert numerisch berechnen

Ausprobieren

```
Remove["Global`*"]
```

```
xL = 1;
```

```
solv = DSolve[{y''[x] == -k x y[x]}, y, x] // Flatten
```

$$\left\{y \to \text{Function}\left[\{x\}, \text{AiryAi}\left[-\frac{k\,x}{(-k)^{2/3}}\right] C[1] + \text{AiryBi}\left[-\frac{k\,x}{(-k)^{2/3}}\right] C[2]\right]\right\}$$

```
Integrate[AiryAi[z],z]
```

$$-\left(z\left(-3\,\text{Gamma}\left[\frac{1}{3}\right]\text{Gamma}\left[\frac{5}{3}\right]\text{HypergeometricPFQ}\left[\left\{\frac{1}{3}\right\},\left\{\frac{2}{3},\frac{4}{3}\right\},\frac{z^3}{9}\right]+3^{1/3}\,z\,\text{Gamma}\left[\frac{2}{3}\right]^2\text{HypergeometricPFQ}\left[\left\{\frac{2}{3}\right\},\left\{\frac{4}{3},\frac{5}{3}\right\},\frac{z^3}{9}\right]\right)\right)\Big/\left(9\,3^{2/3}\,\text{Gamma}\left[\frac{2}{3}\right]\text{Gamma}\left[\frac{4}{3}\right]\text{Gamma}\left[\frac{5}{3}\right]\right)$$

```
Integrate[AiryBi[z],z]
```

$$\left(z\left(3^{2/3}\,\text{Gamma}\left[\frac{1}{3}\right]\text{Gamma}\left[\frac{5}{3}\right]\text{HypergeometricPFQ}\left[\left\{\frac{1}{3}\right\},\left\{\frac{2}{3},\frac{4}{3}\right\},\frac{z^3}{9}\right]+z\,\text{Gamma}\left[\frac{2}{3}\right]^2\text{HypergeometricPFQ}\left[\left\{\frac{2}{3}\right\},\left\{\frac{4}{3},\frac{5}{3}\right\},\frac{z^3}{9}\right]\right)\right)\Big/\left(3\,3^{5/6}\,\text{Gamma}\left[\frac{2}{3}\right]\text{Gamma}\left[\frac{4}{3}\right]\text{Gamma}\left[\frac{5}{3}\right]\right)$$

```
yy[x_] := (y[x] /. solv) /. {C[1] → c1, C[2] → c2}
```

```
yy[x]
```

$$c1\,\text{AiryAi}\left[-\frac{k\,x}{(-k)^{2/3}}\right] + c2\,\text{AiryBi}\left[-\frac{k\,x}{(-k)^{2/3}}\right]$$

```
yy[0]
```

$$\frac{c1}{3^{2/3}\,\text{Gamma}\left[\frac{2}{3}\right]} + \frac{c2}{3^{1/6}\,\text{Gamma}\left[\frac{2}{3}\right]}$$

```
yy[0] // N
```

0.355028 c1 + 0.614927 c2

```
yy[x][[1]][[2]]
```

$\text{AiryAi}\left[-\frac{k\,x}{(-k)^{2/3}}\right]$

```
yy[x][[1]][[2]] /. x → 0
```

$\frac{1}{3^{2/3}\,\text{Gamma}\left[\frac{2}{3}\right]}$

```
yy[x][[2]][[2]]
```

$\text{AiryBi}\left[-\frac{k\,x}{(-k)^{2/3}}\right]$

```
yy[x][[2]][[2]] /. x → 0
```

$\frac{1}{3^{1/6}\,\text{Gamma}\left[\frac{2}{3}\right]}$

```
solve1 = Solve[yy[0] == 0, {c1}] // Flatten
```

$\{c1 \to -\sqrt{3}\ c2\}$

```
yy1[x_] := yy[x] /. solve1; yy1[x]
```

$-\sqrt{3}\ c2\ \text{AiryAi}\left[-\frac{k\,x}{(-k)^{2/3}}\right] + c2\ \text{AiryBi}\left[-\frac{k\,x}{(-k)^{2/3}}\right]$

Nochmals

```
Remove["Global`*"]
```

```
solv = DSolve[{y''[x] == -k x y[x], y[0] == 0}, y, x] // Flatten
```

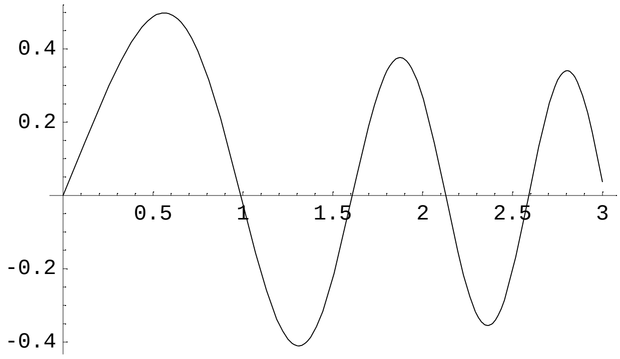
$\left\{y \to \text{Function}\left[\{x\}, -\sqrt{3}\ \text{AiryAi}\left[-\frac{k\,x}{(-k)^{2/3}}\right] C[2] + \text{AiryBi}\left[-\frac{k\,x}{(-k)^{2/3}}\right] C[2]\right]\right\}$

```
solv = DSolve[{y''[x] == -k x y[x], y[0] == 0}, y, x] // Flatten;
yy[x_]:=(y[x]/.solv)/.{C[2]→c2}; yy[x]
```

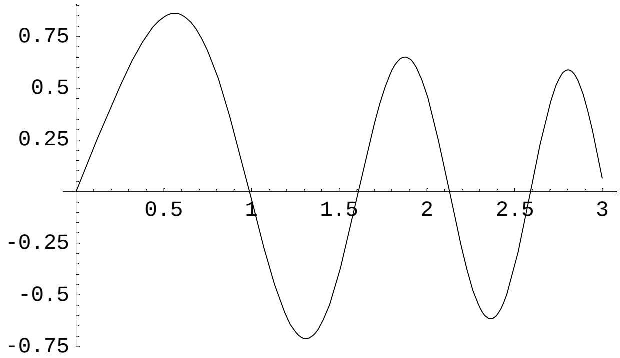
$-\sqrt{3}\ c2\ \text{AiryAi}\left[-\frac{k\,x}{(-k)^{2/3}}\right] + c2\ \text{AiryBi}\left[-\frac{k\,x}{(-k)^{2/3}}\right]$

Bemerkung: Reelle k's erzeugen hier komplexe Nenner im Funktionsargument

```
Plot[Re[(yy[x] /. {c2 → 1, k → 19.613548583248857})], {x, 0, 3}];
```




```
Plot[Im[(yy[x] /. {c2 → 1, k → 19.613548583248857})], {x, 0, 3}];
```




```
FindRoot[Re[(yy[x] /. {c2 → 1, k → 19.613548583248857})], {x, 1}]
```

{x → 0.988702}

```
FindRoot[Im[(yy[x] /. {c2 → 1, k → 19})], {x, 1}]
```

{x → 0.999232}

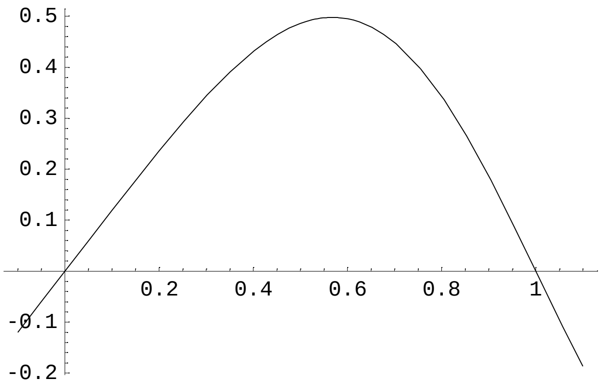
```
FindRoot[Im[(yy[x]/.{c2→1,x→1})],{k,19}]
```

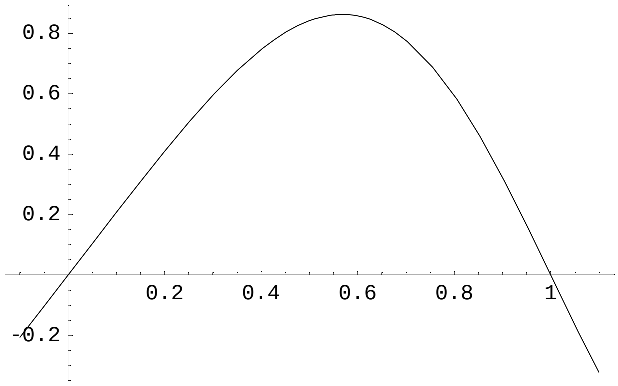
{k → 18.9563}

```
FindRoot[Im[(yy[x]/.{c2→1,k→18.9563})],{x,1}]
```

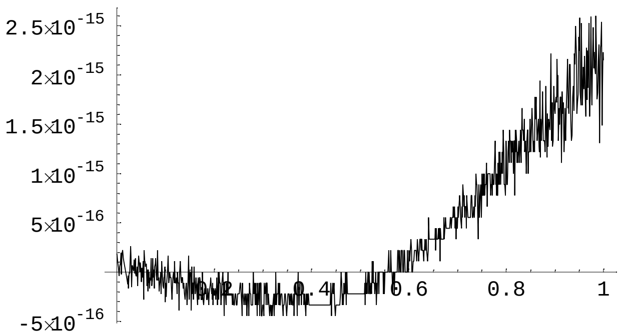
{x → 0.999999}

```
Plot[Re[(yy[x]/.{c2→1,k→18.9563})],{x,-0.1,1.1}];
Plot[Im[(yy[x]/.{c2→1,k→18.9563})],{x,-0.1,1.1}];
```

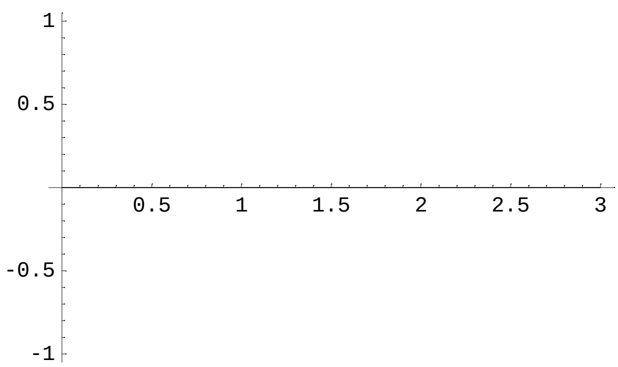





```
Plot[Re[(yy[x]/Re[yy[0.6]]/.{c2→1,k→18.9563})] - Im[(yy[x]/Im[yy[0.6]]/.{c2→1,k→
18.9563})]
,{x,0,1}];
```




```
Plot[Chop[Re[(yy[x]/Re[yy[0.6]]/.{c2→1,k→18.9563})] - Im[(yy[x]/Im[yy[0.6]]/.{c2→
1,k→18.9563})]]
,{x,0,3}];
```



Zum Ritz-Galerkin-Verfahren

```
xL=1;
y1[x_]:=Sin[(Pi)/xL x];
y2[x_]:=x^2 (x-xL)^2;
y3[x_]:=x(x-xL);
A[ya_,yb_]:= -Integrate[D[ya[x],{x,2}] yb[x],{x,0,xL}];
B[ya_,yb_]:= Integrate[x ya[x] yb[x],{x,0,xL}];
H[ya_,yb_,λ_]:= A[ya,yb] - λ B[ya,yb];
m = {
{H[y1,y1,λ],H[y1,y2,λ],H[y1,y3,λ]},
{H[y2,y1,λ],H[y2,y2,λ],H[y2,y3,λ]},
{H[y3,y1,λ],H[y3,y2,λ],H[y3,y3,λ]}
};
Print[m//Simplify//MatrixForm ];
Print[Det[m]//N//ExpandAll//Simplify ];
NSolve[Det[m]==0,{λ}]
```

$$\begin{pmatrix} \frac{1}{4}(2\pi^2-\lambda) & -\frac{2(-12+\pi^2)(2\pi^2-\lambda)}{\pi^5} & \frac{2(-2\pi^2+\lambda)}{\pi^3} \\ -\frac{2(-12+\pi^2)(2\pi^2-\lambda)}{\pi^5} & \frac{24-\lambda}{1260} & \frac{1}{840}(-56+3\lambda) \\ \frac{2(-2\pi^2+\lambda)}{\pi^3} & \frac{1}{840}(-56+3\lambda) & \frac{20-\lambda}{60} \end{pmatrix}$$

$-5.93327\times10^{-14}(-692.818+\lambda)(-178.23+\lambda)(-19.7392+\lambda)$

$\{\{\lambda\to19.7392\},\{\lambda\to178.23\},\{\lambda\to692.818\}\}$

Nochmals: Weitere Eigenwerte

```
Remove["Global`*"]
```

```
solv = DSolve[{y''[x] == -k x y[x], y[0] == 0}, y, x] // Flatten
```

$\left\{y\to\text{Function}\left[\{x\},-\sqrt{3}\,\text{AiryAi}\left[-\frac{k\,x}{(-k)^{2/3}}\right]C[2]+\text{AiryBi}\left[-\frac{k\,x}{(-k)^{2/3}}\right]C[2]\right]\right\}$

```
solv = DSolve[{y''[x] == -k x y[x], y[0] == 0}, y, x] // Flatten;
yy[x_]:=(y[x]/.solv)/.{C[2]→c2}; yy[x]
```

$-\sqrt{3}\,c2\,\text{AiryAi}\left[-\frac{k\,x}{(-k)^{2/3}}\right]+c2\,\text{AiryBi}\left[-\frac{k\,x}{(-k)^{2/3}}\right]$

2. Eigenwert

```
FindRoot[Re[(yy[x]/.{c2→1,x→1})],{k,178.23}]
```

$\{k\to189.221\}$

```
FindRoot[Im[(yy[x]/.{c2→1,x→1})],{k,189.221}]
```

$\{k\to189.221\}$

```
FindRoot[Re[(yy[x]/.{c2→1,k→189.221})],{x,1}]
```

$\{x\to1.\}$

```
FindRoot[Im[(yy[x]/.{c2→1,k→189.221})],{x,1}]
```

{x → 1.}

3. Eigenwert

```
FindRoot[Re[(yy[x]/.{c2→1,x→1})],{k,692.818}]
```

{k → 777.698}

```
FindRoot[Im[(yy[x]/.{c2→1,x→1})],{k,777.698}]
```

{k → 777.698}

```
FindRoot[Re[(yy[x]/.{c2→1,k→777.698})],{x,1}]
```

{x → 1.}

```
FindRoot[Im[(yy[x]/.{c2→1,k→777.698})],{x,1}]
```

{x → 1.}