Lösungen

1. LU-Zerlegung:

(Untersuchung und Herleitung im Falle einer 3 x 3-Matrix)

Das Beispiel einer abstrakten 3 x 3-Matrix:

Konstruktion von U (die Elementarsubstitutionen werden in Matrizenoperationen abgebildet):

Sei a1 nicht 0 (sowie weiter unten auch die Nenner nie 0 - ansonst eine Zeilenvertauschung vorgenommen werden müsste...):

```
Remove["Global`*"]
```

```
A30 = {{a1,b1,c1},{a2,b2,c2},{a3,b3,c3}};
K31 = {{0,0,0},{0,0,0},{A30[[3]][[1]]/A30[[1]][[1]],0,0}};
Map[MatrixForm,{A30, K31}]
```

$$\left\{\begin{pmatrix} a1 & b1 & c1 \\ a2 & b2 & c2 \\ a3 & b3 & c3 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{a3}{a1} & 0 & 0 \end{pmatrix}\right\}$$

```
H13 = IdentityMatrix[3]- K31; H13 // MatrixForm
```

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{a3}{a1} & 0 & 1 \end{pmatrix}$$

```
U1 = H13.A30//Simplify; U1 //MatrixForm
```

$$\begin{pmatrix} a1 & b1 & c1 \\ a2 & b2 & c2 \\ 0 & -\frac{a3\,b1}{a1}+b3 & -\frac{a3\,c1}{a1}+c3 \end{pmatrix}$$

```
K21 = {{0,0,0},{U1[[2]][[1]]/U1[[1]][[1]],0,0},{0,0,0}}; K32 // MatrixForm
```

K32

```
H12 = IdentityMatrix[3]-K21; H12 // MatrixForm
```

$$\begin{pmatrix} 1 & 0 & 0 \\ -\frac{a2}{a1} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

```
U2 = H12.U1//Simplify; U2 //MatrixForm
```

$$\begin{pmatrix} a1 & b1 & c1 \\ 0 & -\frac{a2\,b1}{a1}+b2 & -\frac{a2\,c1}{a1}+c2 \\ 0 & -\frac{a3\,b1}{a1}+b3 & -\frac{a3\,c1}{a1}+c3 \end{pmatrix}$$

Sei $-\frac{a2\,b1}{a1}+b2$ nicht 0:

```
K32 = {{0,0,0},{0,0,0},{0,U2[[3]][[2]]/U2[[2]][[2]],0}}; K32 // MatrixForm
```

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \frac{-\frac{a3\,b1}{a1}+b3}{-\frac{a2\,b1}{a1}+b2} & 0 \end{pmatrix}$$

```
H23 = IdentityMatrix[3]-K32; H23 // MatrixForm
```

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\frac{-\frac{a3\,b1}{a1}+b3}{-\frac{a2\,b1}{a1}+b2} & 1 \end{pmatrix}$$

```
U3 = H23.U2//Simplify; U3 //MatrixForm
```

$$\begin{pmatrix} a1 & b1 & c1 \\ 0 & -\frac{a2\,b1}{a1}+b2 & -\frac{a2\,c1}{a1}+c2 \\ 0 & 0 & \frac{a3\,b2\,c1-a2\,b3\,c1-a3\,b1\,c2+a1\,b3\,c2+a2\,b1\,c3-a1\,b2\,c3}{a2\,b1-a1\,b2} \end{pmatrix}$$

In einem Schritt:

```
U3 = H23.H12.H13.A30//Simplify; U3 //MatrixForm
```

$$\begin{pmatrix} a1 & b1 & c1 \\ 0 & -\frac{a2\,b1}{a1}+b2 & -\frac{a2\,c1}{a1}+c2 \\ 0 & 0 & \frac{a3\,b2\,c1-a2\,b3\,c1-a3\,b1\,c2+a1\,b3\,c2+a2\,b1\,c3-a1\,b2\,c3}{a2\,b1-a1\,b2} \end{pmatrix}$$

```
Uresult = U3;
```

Konstruktion von L mittels inverser Matrix:

L U = A ==> L = A InverseU. Daher ist die Inverse der Dreiecksmatrix U zu bestimmen.
Ansatz: U * InverseU = E. Die Berechnung von InverseU ist hier einfach. Die Sukzessive Berechnung der Elemente von InverseU zeigt rasch, dass InverseU ebenfalls eine obere Dreiecksmatrix sein muss.

```
U = {{u11,u12,u13},{0,u22,u23},{0,0,u33}}; U // MatrixForm
```

$$\begin{pmatrix} u11 & u12 & u13 \\ 0 & u22 & u23 \\ 0 & 0 & u33 \end{pmatrix}$$

```
InvU = {{v11,v12,v13},{0,v22,v23},{0,0,v33}}; InvU // MatrixForm
```

$$\begin{pmatrix} v11 & v12 & v13 \\ 0 & v22 & v23 \\ 0 & 0 & v33 \end{pmatrix}$$

```
U.InvU// MatrixForm
```

$$\begin{pmatrix} u11\,v11 & u11\,v12+u12\,v22 & u11\,v13+u12\,v23+u13\,v33 \\ 0 & u22\,v22 & u22\,v23+u23\,v33 \\ 0 & 0 & u33\,v33 \end{pmatrix}$$

```
InvU.U// MatrixForm
```

$$\begin{pmatrix} u11\,v11 & u12\,v11+u22\,v12 & u13\,v11+u23\,v12+u33\,v13 \\ 0 & u22\,v22 & u23\,v22+u33\,v23 \\ 0 & 0 & u33\,v33 \end{pmatrix}$$

```
solv = Solve[U.InvU == IdentityMatrix[3],{v11,v12,v13,v22,v23,v33}] // Flatten
```

$$\left\{v13\to-\frac{u13\,u22-u12\,u23}{u11\,u22\,u33},\ v11\to\frac{1}{u11},\ v12\to-\frac{u12}{u11\,u22},\ v22\to\frac{1}{u22},\ v23\to-\frac{u23}{u22\,u33},\ v33\to\frac{1}{u33}\right\}$$

Hier hat man ein lineares Gleichungssystem mit 6 Unbekannten, das man nach dem Rückwärtseinsetzungsverfahren rasch und problemlos lösen kann.

```
InvU = InvU /.solv; InvU // MatrixForm
```

$$\begin{pmatrix} \frac{1}{u11} & -\frac{u12}{u11\,u22} & -\frac{u13\,u22-u12\,u23}{u11\,u22\,u33} \\ 0 & \frac{1}{u22} & -\frac{u23}{u22\,u33} \\ 0 & 0 & \frac{1}{u33} \end{pmatrix}$$

Kontrolle:

```
Inverse[U] // MatrixForm
```

$$\begin{pmatrix} \frac{1}{u11} & -\frac{u12}{u11\,u22} & \frac{-u13\,u22+u12\,u23}{u11\,u22\,u33} \\ 0 & \frac{1}{u22} & -\frac{u23}{u22\,u33} \\ 0 & 0 & \frac{1}{u33} \end{pmatrix}$$

Ersetzung der künstlichen Koeffizienten in U durch die von A30:

```
UFlat = Flatten[U];
U3Flat = Flatten[U3];
rul = Table[UFlat[[k]]->U3Flat[[k]],{k,1,Length[UFlat]}]
```

$$\left\{u11\to a1,\ u12\to b1,\ u13\to c1,\ 0\to 0,\ u22\to-\frac{a2\,b1}{a1}+b2,\ u23\to-\frac{a2\,c1}{a1}+c2,\ 0\to 0,\ 0\to 0,\ u33\to\frac{a3\,b2\,c1-a2\,b3\,c1-a3\,b1\,c2+a1\,b3\,c2+a2\,b1\,c3-a1\,b2\,c3}{a2\,b1-a1\,b2}\right\}$$

```
L = A30.InvU; Lresult = L/.rul //Simplify; Lresult // MatrixForm
```

$$\begin{pmatrix} 1 & 0 & 0 \\ \frac{a2}{a1} & 1 & 0 \\ \frac{a3}{a1} & \frac{a3\,b1-a1\,b3}{a2\,b1-a1\,b2} & 1 \end{pmatrix}$$

Kontrolle:

```
Lresult.Uresult //Simplify // MatrixForm
```

$$\begin{pmatrix} a1 & b1 & c1 \\ a2 & b2 & c2 \\ a3 & b3 & c3 \end{pmatrix}$$

```
Simplify[Lresult.Uresult] == A30
```

True

0. Materialbereitstellung

```
A2 = {{1,2},{1,-1}};
B2 = {{3,5},{4,7}};
C2 = {{4,1},{5,6}};
Map[MatrixForm,{A2,B2,C2}]
```

$$\left\{\begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix}, \begin{pmatrix} 3 & 5 \\ 4 & 7 \end{pmatrix}, \begin{pmatrix} 4 & 1 \\ 5 & 6 \end{pmatrix}\right\}$$

```
A3 = {{1,2,3},{1,-1,0},{-2,1,5}};
B3 = {{3,5,1},{4,7,9},{3,2,6}};
C3 = {{4,1,3},{5,6,5},{5,8,8}};
Map[MatrixForm,{A3,B3,C3}]
```

$$\left\{\begin{pmatrix} 1 & 2 & 3 \\ 1 & -1 & 0 \\ -2 & 1 & 5 \end{pmatrix}, \begin{pmatrix} 3 & 5 & 1 \\ 4 & 7 & 9 \\ 3 & 2 & 6 \end{pmatrix}, \begin{pmatrix} 4 & 1 & 3 \\ 5 & 6 & 5 \\ 5 & 8 & 8 \end{pmatrix}\right\}$$

```
A4 = {{1,2,3,4},{1,-1,0,1},{-2,1,5,2},{-2,2,1,5}};
B4 = {{3,5,1,1},{4,7,9,5},{3,2,6,8},{5,6,5,1}};
Map[MatrixForm,{A4,B4}]
```

$$\left\{\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & -1 & 0 & 1 \\ -2 & 1 & 5 & 2 \\ -2 & 2 & 1 & 5 \end{pmatrix}, \begin{pmatrix} 3 & 5 & 1 & 1 \\ 4 & 7 & 9 & 5 \\ 3 & 2 & 6 & 8 \\ 5 & 6 & 5 & 1 \end{pmatrix}\right\}$$

```
A24 = {{3,5,1,1},{4,7,9,5}};
B42 = {{1,2},{1,-1},{-2,1},{-2,2}};
Map[MatrixForm,{A24,B42}]
```

$$\left\{\begin{pmatrix} 3 & 5 & 1 & 1 \\ 4 & 7 & 9 & 5 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 1 & -1 \\ -2 & 1 \\ -2 & 2 \end{pmatrix}\right\}$$

```
X13 = {{x11,x12,x13}};
X31 = {{x11},{x21},{x31}};
X24 = {{x11,x12,x13,x14},{x21,x22,x23,x24}};
X42 = {{x11,x12},{x21,x22},{x31,x32},{x41,x42}};
Map[MatrixForm,{X13,X31,X24,X42}]
```

$$\left\{\begin{pmatrix} x11 & x12 & x13 \end{pmatrix}, \begin{pmatrix} x11 \\ x21 \\ x31 \end{pmatrix}, \begin{pmatrix} x11 & x12 & x13 & x14 \\ x21 & x22 & x23 & x24 \end{pmatrix}, \begin{pmatrix} x11 & x12 \\ x21 & x22 \\ x31 & x32 \\ x41 & x42 \end{pmatrix}\right\}$$

```
b31 = {{50},{-100},{1000}};
b32 = {{50,203},{-100,105},{1000,-50}}; ;
Map[MatrixForm,{b31,b32}]
```

$$\left\{\begin{pmatrix}50\\-100\\1000\end{pmatrix},\begin{pmatrix}50&203\\-100&105\\1000&-50\end{pmatrix}\right\}$$

```
Em2 = IdentityMatrix[2];
Em3 = IdentityMatrix[3];
Em4 = IdentityMatrix[4];
Map[MatrixForm,{Em2,Em3,Em4}]
```

$$\left\{\begin{pmatrix}1&0\\0&1\end{pmatrix},\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix},\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}\right\}$$

```
NullMatrix[m_]:= Table[ Table[0,{k,1,m}],{k,1,m}];
NullMatrix[4]//MatrixForm
```

$$\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}$$

```
ABC = {{a,b,c,d,e},{f,g,h,i,j},{k,l,m,n,o},{p,q,r,s,t},{u,v,w,x,y}}; ABC //
MatrixForm
```

$$\begin{pmatrix}a&b&c&d&e\\f&g&h&i&j\\k&l&m&n&o\\p&q&r&s&t\\u&v&w&x&y\end{pmatrix}$$

```
VdM4 = {{1,2,3,4},{2,3,4,5},{-3,-2,-1,0},{3,4,5,6}}; VdM4 // MatrixForm
```

$$\begin{pmatrix}1&2&3&4\\2&3&4&5\\-3&-2&-1&0\\3&4&5&6\end{pmatrix}$$

1. LU-Zerlegung für 3 x 3 Matrizen als Modul

(Einfach lesbarer Modul ohne kompakte Programmierung, wiederholt anwendbar!)

```
modulLU[{{a1_,b1_,c1_},{a2_,b2_,c2_},{a3_,b3_,c3_}}]:=
Modul[{},
A30 = {{a1,b1,c1},{a2,b2,c2},{a3,b3,c3}};
K31 = {{0,0,0},{0,0,0},{A30[[3]][[1]]/A30[[1]][[1]],0,0}};
Map[MatrixForm,{A30, K31}];
H13 = IdentityMatrix[3]- K31;
U1 = H13.A30//Simplify;
K21 = {{0,0,0},{U1[[2]][[1]]/U1[[1]][[1]],0,0},{0,0,0}};
H12 = IdentityMatrix[3]-K21;
U2 = H12.U1//Simplify;
K32 = {{0,0,0},{0,0,0},{0,U2[[3]][[2]]/U2[[2]][[2]],0}};
H23 = IdentityMatrix[3]-K32;
U3 = H23.U2//Simplify;
Uresult = U3;
U = {{u11,u12,u13},{0,u22,u23},{0,0,u33}};
InvU = {{v11,v12,v13},{0,v22,v23},{0,0,v33}};
solv = Solve[U.InvU == IdentityMatrix[3],{v11,v12,v13,v22,v23,v33}] // Flatten;
InvU = InvU /.solv;
UFlat = Flatten[U];
U3Flat = Flatten[U3];
rul = Table[UFlat[[k]]->U3Flat[[k]],{k,1,Length[UFlat]}];
L = A30.InvU; Lresult = L/.rul //Simplify;
Print["Eingabematrix = ",A30//MatrixForm];
Print["U = ",Uresult//MatrixForm];
Print["L = ",Lresult//MatrixForm];
Print["Kontrolle: L U = ",Lresult.Uresult//MatrixForm];]
```

```
modulLU[A3];
```

$$\text{Eingabematrix} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & -1 & 0 \\ -2 & 1 & 5 \end{pmatrix}$$

$$U = \begin{pmatrix} 1 & 2 & 3 \\ 0 & -3 & -3 \\ 0 & 0 & 6 \end{pmatrix}$$

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -2 & -\frac{5}{3} & 1 \end{pmatrix}$$

$$\text{Kontrolle: L U} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & -1 & 0 \\ -2 & 1 & 5 \end{pmatrix}$$

```
modulLU[B3];
```

$$\text{Eingabematrix} = \begin{pmatrix} 3 & 5 & 1 \\ 4 & 7 & 9 \\ 3 & 2 & 6 \end{pmatrix}$$

$$U = \begin{pmatrix} 3 & 5 & 1 \\ 0 & \frac{1}{3} & \frac{23}{3} \\ 0 & 0 & 74 \end{pmatrix}$$

$$L = \begin{pmatrix} 1 & 0 & 0 \\ \frac{4}{3} & 1 & 0 \\ 1 & -9 & 1 \end{pmatrix}$$

$$\text{Kontrolle: L U} = \begin{pmatrix} 3 & 5 & 1 \\ 4 & 7 & 9 \\ 3 & 2 & 6 \end{pmatrix}$$

```
modulLU[C3];
```

Eingabematrix = $\begin{pmatrix} 4 & 1 & 3 \\ 5 & 6 & 5 \\ 5 & 8 & 8 \end{pmatrix}$

U = $\begin{pmatrix} 4 & 1 & 3 \\ 0 & \frac{19}{4} & \frac{5}{4} \\ 0 & 0 & \frac{47}{19} \end{pmatrix}$

L = $\begin{pmatrix} 1 & 0 & 0 \\ \frac{5}{4} & 1 & 0 \\ \frac{5}{4} & \frac{27}{19} & 1 \end{pmatrix}$

Kontrolle: L U = $\begin{pmatrix} 4 & 1 & 3 \\ 5 & 6 & 5 \\ 5 & 8 & 8 \end{pmatrix}$

2. Determinantenberechnungen

(Berechnung zur Uebung von Hand bis und mit 4 x 4-Matrizen)

```
Det[A2]
```

−3

```
{Det[A2], Det[B2], Det[C2]}
```

{−3, 1, 19}

```
{Det[A3], Det[B3], Det[C3]}
```

{−18, 74, 47}

```
{Det[A4], Det[B4]}
```

{−111, −430}

```
Det[A24]
```

Det::matsq : Argument {{3, 5, 1, 1}, {4, 7, 9, 5}} at position 1 is not a nonempty square matrix. Mehr…

Det[{{3, 5, 1, 1}, {4, 7, 9, 5}}]

Geht nicht!

```
Det[B42]
```

Det::matsq :
 Argument {{1, 2}, {1, −1}, {−2, 1}, {−2, 2}} at position 1 is not a nonempty square matrix. Mehr…

Det[{{1, 2}, {1, −1}, {−2, 1}, {−2, 2}}]

Geht nicht!

Det[X13]

Det::matsq : Argument {{x11, x12, x13}} at position 1 is not a nonempty square matrix. Mehr…

Det[{{x11, x12, x13}}]

Det[X31]

Det::matsq : Argument {{x11}, {x21}, {x31}} at position 1 is not a nonempty square matrix. Mehr…

Det[{{x11}, {x21}, {x31}}]

Det[X24]

Det::matsq :
Argument {{x11, x12, x13, x14}, {x21, x22, x23, x24}} at position 1 is not a nonempty square matrix. Mehr…

Det[{{x11, x12, x13, x14}, {x21, x22, x23, x24}}]

Det[X42]

Det::matsq : Argument {{x11, x12}, {x21, x22}, {x31, x32}, {x41, x42}}
at position 1 is not a nonempty square matrix. Mehr…

Det[{{x11, x12}, {x21, x22}, {x31, x32}, {x41, x42}}]

Det[b31]

Det::matsq : Argument {{50}, {-100}, {1000}} at position 1 is not a nonempty square matrix. Mehr…

Det[{{50}, {-100}, {1000}}]

Det[b32]

Det::matsq :
Argument {{50, 203}, {-100, 105}, {1000, -50}} at position 1 is not a nonempty square matrix. Mehr…

Det[{{50, 203}, {-100, 105}, {1000, -50}}]

{Det[Em2], Det[Em3], Det[Em4], Det[NullMatrix[4]]}

{1, 1, 1, 0}

Det[ABC]

```
eimqu - djmqu - ehnqu + cjnqu + dhoqu - cioqu - eilru + djlru + egnru -
 bjnru - dgoru + bioru + ehlsu - cjlsu - egmsu + bjmsu + cgosu - bhosu -
 dhltu + ciltu + dgmtu - bimtu - cgntu + bhntu - eimpv + djmpv +
 ehnpv - cjnpv - dhopv + ciopv + eikrv - djkrv - efnrv + ajnrv + dforv -
 aiorv - ehksv + cjksv + efmsv - ajmsv - cfosv + ahosv + dhktv - ciktv -
 dfmtv + aimtv + cfntv - ahntv + eilpw - djlpw - egnpw + bjnpw + dgopw -
 biopw - eikqw + djkqw + efnqw - ajnqw - dfoqw + aioqw + egksw - bjksw -
 eflsw + ajlsw + bfosw - agosw - dgktw + biktw + dfltw - ailtw - bfntw +
 agntw - ehlpx + cjlpx + egmpx - bjmpx - cgopx + bhopx + ehkqx -
 cjkqx - efmqx + ajmqx + cfoqx - ahoqx - egkrx + bjkrx + eflrx -
 ajlrx - bforx + agorx + cgktx - bhktx - cfltx + ahltx + bfmtx -
 agmtx + dhlpy - cilpy - dgmpy + bimpy + cgnpy - bhnpy - dhkqy +
 cikqy + dfmqy - aimqy - cfnqy + ahnqy + dgkry - bikry - dflry +
 ailry + bfnry - agnry - cgksy + bhksy + cflsy - ahlsy - bfmsy + agmsy
```

Det[VdM4]

0